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THE (0,2) EXACTLY SOLVABLE STRUCTURE OF CHIRAL RINGS, LANDAU-GINZBURG THEORIES AND CALABI-YAU MANIFOLDS

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ABSTRACT

We identify the exactly solvable theory of the conformal fixed point of (0,2) Calabi-Yau σ -models and their Landau-Ginzburg phases. To this end we consider a number of (0,2) models constructed from a particular (2,2) exactly solvable theory via the method of simple currents. In order to establish the relation between exactly solvable (0,2) vacua of the heterotic string, (0,2) Landau-Ginzburg orbifolds and (0,2) Calabi-Yau manifolds, we compute the Yukawa couplings of the exactly solvable model and compare the results with the product structure of the chiral ring which we extract from the structure of the massless spectrum of the exact theory. We find complete agreement between the two up to a finite number of renormalizations. For a particularly simple example we furthermore derive the generating ideal of the chiral ring from a (0,2) linear σ -model which has both a Landau-Ginzburg and a (0,2) Calabi-Yau phase.

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1 Introduction

Ever since the revival of superstring theory ten years ago (2,2) supersymmetric vacua have taken center stage. This has been the case despite the fact that the main motivation for considering supersymmetric groundstates, the hierarchy problem, actually necessitates only the consideration of (0,2) conformal field theories [1]. The reason for this focus on an apparently rather special class of theories is to be found not so much in the fact that (0,2) models were somewhat hard to come by. Indeed, (0,2) models have been constructed a number of years ago by several authors [2, 3, 4, 5, 6]¹. More important is the fact that (2,2) theories are of phenomenological interest, leading to three-generation models with small gauge groups at the Planck scale [8, 9] and that they raise a number of fundamental and intriguing questions. A short and incomplete list of such interesting aspects might comprise the possibility of conifold phase transitions [10, 11] in the early universe, the exact solution of Calabi–Yau σ -models [12, 13, 14, 15] as well as the fact that the moduli space features a number of important properties, such as special geometry [16], world sheet mirror symmetry [17, 18], spacetime mirror symmetry [19, 20] and scaling [21]. The full understanding of these problems has been slow to emerge and most of them have not yet been resolved in a completely satisfactory fashion even in the simpler context of (2,2) theories.

The fact remains, however, that (0,2) theories do provide a phenomenologically appealing framework [22] and a tantalizing question has been for some time what the generic features of the space of (0,2) vacua are and which, if any, of the features mentioned above survive in this more general context. Part of the problem has been that even though exactly solvable (0,2) models have been known for many years, their relation to (0,2) σ -models has remained obscure. Unlike the situation in the framework of (2,2) compactifications, where the conformal fixed points of particularly simple types of Calabi–Yau σ -models have been shown [13, 23, 24, 25, 9][26, 27, 28] to be described by Gepner models [12] or Kazama–Suzuki models [29, 30, 31, 32], no such understanding has emerged in the context of (0,2) theories. It is this question of the existence and nature of the exact theory describing the conformal fixed points of (0,2) σ -models which we address in the present paper.

In [15] Witten formulated a framework which has been employed in [33] to formulate a class of (0,2) supersymmetric Landau–Ginzburg models, generalizing the construction of the class of (2,2) Landau–Ginzburg theories [14, 34]. This Landau–Ginzburg formulation allows us to address the problem of destabilization of (0,2) σ -models [35, 36, 37, 38, 39, 40] and facilitates the computation of some important characteristics of these theories, but falls short of addressing the important problem of identifying the exactly solvable structure of the superconformal fixed points. In [41] a

¹More recently (0,2) exactly solvable theories based on coset theories have been considered in [7].

class of exactly solvable $(0,2)$ models with $(c, \bar{c}) = (6 + r, 9)$, $r = 3, 4, 5$ and gauge group of rank $(9 - r)$ was constructed by generating new modular invariants from the class of Gepner models via a slight modification of the simple current method of [42, 6]. By considering simple currents which break part of the supersymmetry one finds that in general there exist several $(0,2)$ daughter theories which can be built from any of the Gepner models. The resulting class of theories thus is much richer than the original class of $(2,2)$ tensor models. Furthermore, even for theories with an E_6 gauge group one obtains spectra which do not appear among the class of Gepner models [12], the complete list of which has been constructed in [26, 27, 28]². This supports the expectation that $(2,2)$ theories describe but a small part of the total space of all vacua with $N=1$ spacetime supersymmetry.

In the present paper we generalize the $(2,2)$ triality of exactly solvable models, Landau–Ginzburg theories and Calabi–Yau manifolds, to the context of $(0,2)$ string vacua by establishing a relation between the linear σ –models considered in [33] and the exactly solvable models of [41]. The technique we use is reminiscent of the construction introduced in [18]. It was shown there that for a particular class of discrete symmetries acting on a $(2,2)$ exactly solvable model or its Landau–Ginzburg counterpart, it is possible not only to derive the anomalous dimensions of the chiral primary fields of the orbifold, but also to find a superpotential which describes the orbifold theory. The implementation of simple currents is similar to the action of discrete symmetries in that they also change the dimensions of the original fields. We show that by considering the detailed structure of the massless spectrum it is possible to extract the anomalous dimensions of the scaling fields of the $(0,2)$ theory and to derive the chiral ring describing the $(0,2)$ theory. We furthermore show that our identification is correct by computing the Yukawa couplings of the exactly solvable models as well as the product structure of the chiral ring. Comparing the two emerging patterns leads to perfect agreement up to a change of basis.

The paper is organized as follows. In Section 2 we review the basic ingredients of the construction of [41] before we proceed in Sections 3 and 4 to establish in some detail the relation between a particularly simple exactly solvable $(0,2)$ model and a linear $(0,2)$ σ –model. To this end we first determine in Section 3 the massless spectrum and compute the Yukawa couplings in the exact theory. In Section 4 we derive the chiral ring from the structure of the generations of the exact model and analyze its product structure, the comparison of which with the exact Yukawa

²They do appear in the class of all Landau–Ginzburg theories [43, 44] and therefore might describe $(0,2)$ deformations of known $(2,2)$ theories. The complete list of Landau–Ginzburg theories can be accessed on the web at the Calabi–Yau pages at <http://www.math.okstate.edu/~katz/CY> and its European mirror <http://thew02.physik.uni-bonn.de/~netah/cy.html>. (These pages are in an experimental stage and prone to changes.)

couplings leads to complete agreement. We close this Section by showing that the resulting chiral ring can be derived from a particular (0,2) linear σ -model defining a stable bundle over a smooth Calabi–Yau manifold. This linear σ -model features both a Landau–Ginzburg phase and a (0,2) Calabi–Yau phase and therefore the exactly solvable model described in Section 3 indeed provides the underlying exact conformal field theory of a (0,2) Calabi–Yau manifold at a particular point in the moduli space. It then follows from the work of [39, 40] that the theories contained in a neighborhood of this point in moduli space are also conformally invariant. In the remaining Sections we extend our considerations to further models.

2 Exactly solvable theories

2.1 New models from old via simple currents

The class of exactly solvable models which we will focus on has been described in some detail in [41]. In order to make the present article self-contained we begin by reviewing the salient features of the construction.

The basic tool for the construction is the simple current technique [42, 6] for building new modular invariants from old ones. Briefly, one considers a rational conformal field theory with a given modular invariant partition function which is supposed to contain a simple current J of index N and monodromy parameter R , i.e. a unipotent field ($J^N = 1$) such that for primary fields Φ_i of the theory

$$J \times \Phi_i = \Phi_j, \quad \Delta(J) = \frac{R(N-1)}{2N} \pmod{1}. \quad (2.1.1)$$

R then is defined modulo N for N odd and modulo $2N$ for N even. Furthermore one introduces a monodromy charge Q_J for primary fields associated with a given simple current J

$$Q_J(\Phi_i) = \Delta(\Phi_i) + \Delta(J) - \Delta(J \cdot \Phi_i) \pmod{1}, \quad (2.1.2)$$

which takes values $\frac{t}{N}$, $t \in \mathbb{Z}$. Of importance to the construction will be a slightly modified monodromy charge defined on the element of each orbit by

$$\hat{Q}(J^p \Phi_i) = \frac{t + pR}{2N} \pmod{1}. \quad (2.1.3)$$

The simple current and its iterative application defines orbits of all the primary fields $\Phi_i, J \Phi_i, \dots, J^d \Phi_i$, where d is a divisor of N . If R is even the matrix

$$M_{ij}(J) := \delta^1 \left(\hat{Q}(\Phi_i) + \hat{Q}(\Phi_j) \right) \sum_{p=1}^N \delta(\Phi_i, J^p \Phi_j) \quad (2.1.4)$$

with $\delta^1(x) = 1$ for $x \in \mathbb{Z}$ and zero otherwise, defines a new modular invariant partition function

$$Z(\tau, \bar{\tau}) = \sum_{i,j} \chi_i(\tau) M_{ij} \chi_j(\bar{\tau}). \quad (2.1.5)$$

This procedure allows for an iteration procedure with a whole bunch of simple currents by considering

$$Z(\tau, \bar{\tau}) \sim \vec{\chi}(\tau) M(J_n) \cdots M(J_2) M(J_1) \vec{\chi}(\bar{\tau}), \quad (2.1.6)$$

where \sim indicates equality up to an overall factor originating from a universal multiplicity factor.

2.2 (0,2) Simple current modular invariants leading to E_{9-r}

We now wish to apply the above general considerations to the particular case, where the final theory we end up with has the following properties: a) The gauge group is any of the groups $E_{9-r} \ni \{E_6, E_5 = \text{SO}(10), E_4 = \text{SU}(5), E_3 = \text{SU}(3) \times \text{SU}(2)\}$. b) The central charges in the two internal sectors are $(c, \bar{c}) = (6 + r, 9)$. c) Besides the right moving $U(1)_R$ current which is part of the right moving N=2 superconformal theory, there exists also a left moving $U(1)_L$ current J_L satisfying the operator product expansion

$$J_L(z) J_L(w) = \frac{r}{(z-w)^2} + \text{reg.} \quad (2.2.1)$$

d) Only the subset $\text{SO}(16-2r) \times U(1)_L \subset E_{9-r}$ of the gauge group is linearly realized, the full E_{9-r} being generated by taking orbits with respect to the spectral flow operator of conformal dimension $(\Delta, Q) = (\frac{r}{8}, \frac{r}{2})$. It is convenient to describe the construction in the following left-right symmetric way, where the asymmetry between the left and right sector is achieved at the end of the day by throwing away part of the right moving current algebra in such a way as not to endanger modular invariance. Consider then an internal conformal field theory with the ingredients in Table 1.

	Left Sector c	Right Sector \bar{c}
N=2 SCFT	9	9
$(U(1)_2)^{r-3}$	$r-3$	$r-3$
$\text{SO}(16-2r) \times E_8$	$16-r$	$16-r$

Table 1: *Ingredients for the construction of the internal theory.*

The crucial, new ingredient, as compared to Gepner's tensor model construction, is the free boson ϕ compactified on a circle of radius $R=2$, denoted by $U(1)_2$. The diagonal partition function of this theory is simply

$$Z_{U(1)_2}(\tau, \bar{\tau}) = \sum_{m=-1}^2 \Theta_{m,2}(\tau) \Theta_{m,2}(\bar{\tau}). \quad (2.2.2)$$

The conformal dimension and charge of the primary fields $\Phi_{m,2}^{U(1)_2}$ are

$$(\Delta, Q) (\Phi_{m,2}^{U(1)_2}) = \left(\frac{m^2}{8}, \frac{m}{2} \right). \quad (2.2.3)$$

The current $j_{U(1)_2} = i\partial_z\phi$ satisfies the following OPE

$$j_{U(1)_2}(z)j_{U(1)_2}(w) = \frac{1}{(z-w)^2} + \text{reg}. \quad (2.2.4)$$

Consider first the left moving sector. Even though the $U(1)_2$ theory is not N=2 supersymmetric, it does feature a spectral flow between the m even sectors and the m odd sectors. Since this boson describes a Dirac fermion the spectral flow operator $e^{i\phi(z)/2}$, with conformal dimension and charge $(\frac{1}{8}, \frac{1}{2})$, relates the NS sector and the R sector of the Dirac fermion. The introduction of the $U(1)_2$ current algebra is motivated by the fact that by combining its current $j_{U(1)_2}$ with the $U(1)$ current $j_{c=9} = i\sqrt{3}\partial\Phi$ of the N=2, c=9 theory one obtains the OPE (2.2.1) for $U(1)_L$. Furthermore, putting together the spectral flow operators of the N=2 conformal field theory with the $U(1)_2$ one obtains the spectral flow operator of the $c = (6+r)$ left moving sector

$$\Sigma_{c=6+r}(z) = e^{i\frac{\sqrt{3}}{2}\Phi(z)} \cdot \prod_{j=1}^{r-3} e^{\frac{i}{2}\phi_j(z)}. \quad (2.2.5)$$

It can be shown that this operator does indeed extend the group $SO(16-2r) \times U(1)_L$ to E_{9-r} .

Before turning to the right moving sector, it is useful to review a few facts about the representations of $SO(2n)$ Kac-Moody algebras at level $k = 1$. Recall that the representations of an $SO(2n)$ algebra decompose into scalars (0), spinors (s), antispinors (c) and vectors (v). The characters and the quantum numbers of the corresponding primary fields are collected in Table 2.

Character	Conformal Dimension Δ	$Q \bmod 2$	Degeneracy
$\chi_0 = \frac{1}{2} \left(\left(\frac{\vartheta_3}{\eta} \right)^n + \left(\frac{\vartheta_4}{\eta} \right)^n \right)$	0	0	1
$\chi_v = \frac{1}{2} \left(\left(\frac{\vartheta_3}{\eta} \right)^n - \left(\frac{\vartheta_4}{\eta} \right)^n \right)$	$\frac{1}{2}$	1	$2n$
$\chi_s = \frac{1}{2} \left(\left(\frac{\vartheta_2}{\eta} \right)^n + \left(\frac{\vartheta_1}{\eta} \right)^n \right)$	$\frac{n}{8}$	$\frac{n}{2}$	2^{n-1}
$\chi_c = \frac{1}{2} \left(\left(\frac{\vartheta_2}{\eta} \right)^n - \left(\frac{\vartheta_1}{\eta} \right)^n \right)$	$\frac{n}{8}$	$\frac{n}{2} - 1$	2^{n-1}

Table 2: Characters of $SO(2n)$. The charge Q is taken with respect to the sum of all Cartan elements of the Lie algebra and ϑ_i denote the Jacobi theta functions.

Because our goal is to eventually turn the bosonic theory into a heterotic theory by applying the bosonic string map as considered by Gepner [12]

$$\begin{aligned} \chi_0^{SO(10) \times E_8} &\longrightarrow \chi_v^{SO(2)} & \chi_v^{SO(10) \times E_8} &\longrightarrow \chi_0^{SO(2)} \\ \chi_s^{SO(10) \times E_8} &\longrightarrow -\chi_c^{SO(2)} & \chi_c^{SO(10) \times E_8} &\longrightarrow -\chi_s^{SO(2)}, \end{aligned} \quad (2.2.6)$$

we wish to extend the $\text{SO}(16-2r) \times \text{U}(1)_2^{r-3}$ to $\text{SO}(10)$. In the right sector this is achieved by considering $(r-3)$ simple currents of the form

$$\mathbf{J}_{\text{ext}} = \Phi_{2,2}^{\text{U}(1)_2} \otimes \Phi_v^{\text{SO}(16-2r)}, \quad (2.2.7)$$

with $\Delta(\mathbf{J}_{\text{ext}}) = 1$, which generate orbits of $\text{SO}(10)$. For $r = 4$, say, these orbits take the form

$$\begin{aligned} \chi_0^{\text{SO}(10)} &= \chi_0^{\text{SO}(8)} \Theta_{0,2} + \chi_v^{\text{SO}(8)} \Theta_{2,2} & \chi_v^{\text{SO}(10)} &= \chi_0^{\text{SO}(8)} \Theta_{2,2} + \chi_v^{\text{SO}(8)} \Theta_{0,2} \\ \chi_s^{\text{SO}(10)} &= \chi_s^{\text{SO}(8)} \Theta_{1,2} + \chi_c^{\text{SO}(8)} \Theta_{-1,2} & \chi_c^{\text{SO}(10)} &= \chi_c^{\text{SO}(8)} \Theta_{0,2} + \chi_s^{\text{SO}(8)} \Theta_{2,2}. \end{aligned} \quad (2.2.8)$$

So far our considerations have been completely general. From this point on we focus our attention on products of N=2 minimal tensor models a la Gepner.

2.3 (0,2) Theories from Gepner models

Recall that for the minimal models the conformal anomaly is

$$c = \frac{3k}{k+2} \quad (2.3.1)$$

and the dimensions and charges of the chiral primary fields $\Phi^{l,q,s}$ are given by

$$\Delta^{l,q,s} = \frac{l(l+2) - q^2}{4(k+2)} + \frac{s^2}{8}, \quad Q^{l,q,s} = -\frac{q}{k+2} + \frac{s}{2}, \quad (2.3.2)$$

where the level $k \in \mathbb{N}$, $0 \leq l \leq k$, $l - q + s \in 2\mathbb{Z}$ and $|q - s| \leq l$. Here the range of the various quantum numbers is $l = 0, \dots, k$, $q \sim q + 2(k+2)$, $s \sim s + 4$ and we will employ the following notation for the complete fields

$$\Phi_{l,\bar{q},\bar{s}}^{l,q,s} = \begin{bmatrix} l & q & s \\ \bar{l} & \bar{q} & \bar{s} \end{bmatrix}, \quad (2.3.3)$$

which, if $l = \bar{l}$, simplifies into

$$\Phi_{l,\bar{q},\bar{s}}^{l,q,s} = \begin{bmatrix} l & q & s \\ l & \bar{q} & \bar{s} \end{bmatrix}. \quad (2.3.4)$$

Important, finally, is the identification

$$\begin{bmatrix} l & q & s \\ \bar{l} & \bar{q} & \bar{s} \end{bmatrix} \sim \begin{bmatrix} k-l & q+(k+2) & s+2 \\ k-\bar{l} & \bar{q}+(k+2) & \bar{s}+2 \end{bmatrix}. \quad (2.3.5)$$

In [41] the tensor models $\bigotimes_{i=1}^n k_i$ have been used to construct (0,2) theories with the aforementioned properties a) – d) by breaking the left moving supersymmetry as well as the E_6 gauge group present in Gepner's construction. The following steps do the job:

- The simple currents extending $\text{SO}(16-2r) \times \text{U}(1)_2^{r-3}$ to $\text{SO}(10)$ in the tensor model can be written as

$$J_{\text{ext}}^j = [0 \ 0 \ 0]^{\otimes n} \otimes [0]^{j-1} [2] [0]^{r-3-j} \otimes [v], \quad j = 1, \dots, r-3, \quad (2.3.6)$$

where $[v]$ denotes the vector representation of $\text{SO}(16-2r)$.

- Next one has to make sure, as usual, that only fields of same character couple, i.e. Neveu-Schwarz ($s \in \{0, 2\}$) to Neveu-Schwarz and Ramond ($s \in \{1, 3\}$) to Ramond. This is achieved via the projection operators $J_i = G_i \otimes \Phi_v^{\text{SO}(16-2r)}$, where the $G_i = \Phi^{k, k+2, 4}$ are the supercurrents in the i 'th factor of the tensor model. The appropriate fields in a tensor model with n factors take the (chiral) form

$$J_i = [0 \ 0 \ 0]^{\otimes(i-1)} \otimes [k \ (k+2) \ 4] \otimes [0 \ 0 \ 0]^{\otimes(n-i)} \otimes [0]^{r-3} \otimes [v], \quad i = 1, \dots, n. \quad (2.3.7)$$

- The next step is to implement the right moving GSO projection onto states with even overall charge. This is achieved via the simple current

$$\bar{J}_{\text{GSO}} = \bar{\Sigma}_{c=9} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}^{\otimes(r-3)} \otimes \begin{bmatrix} 0 \\ s \end{bmatrix}, \quad (2.3.8)$$

where

$$\bar{\Sigma}_{c=9} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{\otimes n} \quad (2.3.9)$$

is the right moving spectral flow operator of dimensions $(\Delta, \bar{\Delta})(\bar{\Sigma}_{c=9}) = (0, \frac{3}{8})$ and charges $(Q, \bar{Q})(\bar{\Sigma}_{c=9}) = (0, \frac{3}{2})$. Implementing these projections leads to the partition function

$$Z \sim \bar{\chi}(\tau) M(\bar{J}_{\text{GSO}}) \left(\prod_{i=1}^n M(J_i) \right) \left(\prod_{j=1}^{r-3} M(J_{\text{ext}}^j) \right) \bar{\chi}(\bar{\tau}). \quad (2.3.10)$$

At this point all the conditions for a (2,2) supersymmetric theory have been implemented. After turning this left-right symmetric theory into a class of heterotic string vacua by applying the bosonic string map in the way described above, we should expect the procedure described thus far to provide an alternative construction of Gepner's models. The fact that the partition function (2.3.10) indeed reproduces the expected spectra provides a nice check of the implementation.

- Our goal, however, was to obtain an asymmetric (0,2) CFT and in order to do so one simply has to introduce further simple currents in the left moving sector, denoted by Υ_i in the following, which do not commute with the simple currents that appear in (2.3.10). Of particular interest are simple currents Υ_i which break both the left moving $\text{N}=2$ supersymmetry and the E_6 gauge group which results from the \bar{J}_{GSO} projection. It turns out that for

each tensor model one can find a multitude of such simple currents, making this class an interesting framework. Given the existence of fields Υ_i with the appropriate properties the only remaining ingredient is the left moving GSO projection which is implemented by the simple current

$$J_{\text{GSO}} = \Sigma_{c=6+r} \otimes \begin{bmatrix} s \\ 0 \end{bmatrix}, \quad (2.3.11)$$

where

$$\Sigma_{c=6+r} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}^{\otimes n} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\otimes(r-3)}. \quad (2.3.12)$$

The crucial effect of the simple currents Υ_i is that they prevent the extension of the gauge group to E_6 via the currents J_i and J_{ext} . Instead the chiral spectral flow is $\Sigma_{c=6+r}^2$ and thus is an operator of dimension $(\Delta, \bar{\Delta}) = (\frac{r}{2}, 0)$ and charges $(Q, \bar{Q}) = (r, 0)$. It is this operator which extends the $SO(16-2r) \times U(1)_L$ to E_{9-r} .

Putting everything together then leads to the final form of the partition function [41]

$$Z \sim \bar{\chi}(\tau) M(J_{\text{GSO}}) \left(\prod_i M(\Upsilon_i) \right) M(\bar{J}_{\text{GSO}}) \left(\prod_{i=1}^r M(J_i) \right) \left(\prod_{j=1}^{r-3} M(J_{\text{ext}}^j) \right) \bar{\chi}(\bar{\tau}). \quad (2.3.13)$$

This partition function exhibits all the desired features in order to be of use in the exploration of exactly solvable models which possibly describe particular points in the moduli space of Landau–Ginzburg theories constructed in [33]. In short one might summarize the structure of these vacua as described in Table 3.

	Left Sector	Right Sector
Ghosts b, c	−26	−26
Super Ghosts β, γ	−	11
Spacetime Coordinates X^μ	4	4
Superpartners ψ^μ of X^μ	−	2
Internal CFT	$6+r$	9
Gauge Group $SO(16-2r) \times E_8$	$16-r$	−

Table 3: *Anomaly structure of the complete (0,2) theory.*

3 An (80,0) SO(10) (0,2)–model

The exact theories we focus on in the following are all derived from the parent ‘quintic’ tensor model defined by considering the product of five N=2 minimal factors at level $k=3$. We will denote

this theory by $3^{\otimes 5}$. The different spectra and gauge groups will be obtained by applying various simple currents Υ_i .

3.1 The massless spectrum

Our first exact model is based on the simple current

$$\Upsilon_1 = [3 \ 0 \ -1] \otimes [0 \ 0 \ 0]^{\otimes 4} \otimes [1] \otimes [0], \quad (3.1.1)$$

which only affects the first of the five minimal $N=2$ factors and turns out to break the E_6 of the parent Gepner model down to $SO(10)$. The spectrum therefore is arranged into representations of $SO(10)$ and we have summarized the relevant multiplicities of the massless sector in Table 4.

SO(10) Representation:	0	10	16	$\overline{16}$
Spin 0:	350	74	80	0
Spin 1:	7	0	0	0

Table 4: *Massless spectrum of the $(80,0)$ $SO(10)$ daughter of the $3^{\otimes 5}$ model.*

Of particular relevance for the following are the generations, which can be represented in a number of ways. Considering the internal part of a generation only, the **16** of the $SO(10)$ in the (-1) ghost picture as a spacetime scalar leads to a vertex operator of the form (at zero momentum)

$$V_{-1}^a = e^{-\rho(\bar{z})} \mathcal{O}_{16}(z, \bar{z}) \lambda^a, \quad (3.1.2)$$

where the internal operator $\mathcal{O}_{16}(z, \bar{z})$ has $U(1)$ charge $(Q, \bar{Q}) = (-1, -1)$, and $\rho(\bar{z})$ is the bosonized supersymmetry ghost, whereas the fermions λ^a generate the $SO(10)$ current algebra. The operator has $\Delta = \bar{\Delta} = \frac{1}{2}$ and therefore defines a chiral primary field. In this representation the generations take the form exhibited in Table 5, in which we have used the abbreviations

$$g_0 = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix}, \quad g_1 = \begin{bmatrix} 2 & -3 & -1 \\ 2 & 2 & 0 \end{bmatrix} \quad (3.1.3)$$

for the two states that appear in the first minimal factor and

$$u_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad u_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad (3.1.4)$$

for the $U(1)$ -part of the generations.

Type	Field	Number
I	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \end{bmatrix} u_0 \begin{bmatrix} v \\ v \end{bmatrix}$	12
II	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \end{bmatrix} u_0 \begin{bmatrix} v \\ v \end{bmatrix}$	12
III	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} u_0 \begin{bmatrix} v \\ v \end{bmatrix}$	12
IV	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} u_0 \begin{bmatrix} v \\ v \end{bmatrix}$	4
V_i	$g_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \end{bmatrix} u_i \begin{bmatrix} v \\ v \end{bmatrix}$	8
VI_i	$g_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} u_i \begin{bmatrix} v \\ v \end{bmatrix}$	24
VII_i	$g_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} u_i \begin{bmatrix} v \\ v \end{bmatrix}$	8

Table 5: Generations of the $(80,0)$ $SO(10)$ daughter of $3^{\otimes 5}$.

The **10** of $SO(10)$ can be counted by enumerating the singlet part of the decomposition $\mathbf{10} = \mathbf{1}_{-2} \oplus \mathbf{8}_0^s \oplus \mathbf{1}_2$ of the vectors of $SO(10)$ with respect to the maximal subgroup $SO(8) \times U(1)$ which, using abbreviations

$$v_0 = \begin{bmatrix} -2 & 0 \\ -4 & -2 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 3 & 1 \\ -4 & -2 \end{bmatrix} \quad (3.1.5)$$

for the relevant primary fields in the first factor and u_i^+ for the charge conjugate of u_i , leads to the list of vectors contained in Table 6.

Type	Field	Number
A	$\begin{bmatrix} 0 & 0 & 0 \\ -2 & -2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -3 & -3 & -2 \end{bmatrix} \begin{bmatrix} 3 & -3 & 0 \\ -5 & -5 & -2 \end{bmatrix} \begin{bmatrix} 3 & -3 & 0 \\ -5 & -5 & -2 \end{bmatrix} \begin{bmatrix} 3 & -3 & 0 \\ -5 & -5 & -2 \end{bmatrix} u_0 \begin{bmatrix} 0 \\ v \end{bmatrix}$	4
B	$\begin{bmatrix} 0 & 0 & 0 \\ -2 & -2 & -2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ -4 & -4 & -2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ -4 & -4 & -2 \end{bmatrix} \begin{bmatrix} 3 & -3 & 0 \\ -5 & -5 & -2 \end{bmatrix} \begin{bmatrix} 3 & -3 & 0 \\ -5 & -5 & -2 \end{bmatrix} u_0 \begin{bmatrix} 0 \\ v \end{bmatrix}$	6
C_i	$v_i \begin{bmatrix} 0 & 0 & 0 \\ -2 & -2 & -2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ -4 & -4 & -2 \end{bmatrix} \begin{bmatrix} 3 & -3 & 0 \\ -5 & -5 & -2 \end{bmatrix} \begin{bmatrix} 3 & -3 & 0 \\ -5 & -5 & -2 \end{bmatrix} u_i^+ \begin{bmatrix} 0 \\ v \end{bmatrix}$	24
D_i	$v_i \begin{bmatrix} 1 & -1 & 0 \\ -3 & -3 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -3 & -3 & -2 \end{bmatrix} \begin{bmatrix} 3 & -3 & 0 \\ -5 & -5 & -2 \end{bmatrix} \begin{bmatrix} 3 & -3 & 0 \\ -5 & -5 & -2 \end{bmatrix} u_i^+ \begin{bmatrix} 0 \\ v \end{bmatrix}$	12
E_i	$v_i \begin{bmatrix} 1 & -1 & 0 \\ -3 & -3 & -2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ -4 & -4 & -2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ -4 & -4 & -2 \end{bmatrix} \begin{bmatrix} 3 & -3 & 0 \\ -5 & -5 & -2 \end{bmatrix} u_i^+ \begin{bmatrix} 0 \\ v \end{bmatrix}$	24
F_i	$v_i \begin{bmatrix} 2 & -2 & 0 \\ -4 & -4 & -2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ -4 & -4 & -2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ -4 & -4 & -2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ -4 & -4 & -2 \end{bmatrix} u_i^+ \begin{bmatrix} 0 \\ v \end{bmatrix}$	2
G_1	$\begin{bmatrix} 2 & 4 & 2 \\ -4 & -4 & -2 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ -4 & -4 & -2 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ -4 & -4 & -2 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ -4 & -4 & -2 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ -4 & -4 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ v \end{bmatrix}$	1
G_2	$\begin{bmatrix} 2 & -1 & 1 \\ -4 & -4 & -2 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ -4 & -4 & -2 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ -4 & -4 & -2 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ -4 & -4 & -2 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ -4 & -4 & -2 \end{bmatrix} u_1^+ \begin{bmatrix} 0 \\ v \end{bmatrix}$	1

Table 6: Vectors of the $SO(10)$ $(80,0)$ daughter of $3^{\otimes 5}$.

With the explicit form of the massless spectrum at our disposal we can now explore this model in greater depth by computing the Yukawa couplings. This is what we will turn to in the next Subsection.

3.2 The exact Yukawa couplings

Suppose we wish to compute the Yukawa couplings

$$< \mathbf{10} \cdot \mathbf{16} \cdot \mathbf{16} > . \quad (3.2.1)$$

Then we have to specify the vertex operators and the picture in which they live. To get the ghostnumber -2 of string theory tree level correlation functions [45] one might consider vertex operators

$$< V_{-1}^{\mathbf{10}} V_{-1/2}^{\mathbf{16}} V_{-1/2}^{\mathbf{16}} > . \quad (3.2.2)$$

Using the decompositions

$$\mathbf{10} = \mathbf{1}_{-2} \oplus \mathbf{8}_0^s \oplus \mathbf{1}_2, \quad \mathbf{16} = \mathbf{8}_{-1}^v \oplus \mathbf{8}_1^c \quad (3.2.3)$$

of the $\text{SO}(10)$ representations with respect to the maximal subgroup $\text{SO}(8) \times \text{U}(1) \subset \text{SO}(10)$, the couplings $< \mathbf{10} \cdot \mathbf{16} \cdot \mathbf{16} >$ decompose into

$$\left\langle \begin{pmatrix} \mathbf{1}_{-2} \\ \mathbf{8}_0^s \\ \mathbf{1}_2 \end{pmatrix} \begin{pmatrix} \mathbf{8}_{-1}^v \\ \mathbf{8}_1^c \end{pmatrix} \begin{pmatrix} \mathbf{8}_{-1}^v \\ \mathbf{8}_1^c \end{pmatrix} \right\rangle = < \mathbf{1}_2 \cdot \mathbf{8}_{-1}^v \cdot \mathbf{8}_{-1}^v > = < \mathbf{8}_0^s \cdot \mathbf{8}_1^c \cdot \mathbf{8}_{-1}^v > . \quad (3.2.4)$$

The sum over all right moving charges also has to vanish, a condition which is not satisfied for the fields listed in Tables 5 and 6. Using the supersymmetry charge operator $\bar{\Sigma}_{c=9}$ (2.3.9) of the Gepner parent model one obtains the desired form

$$V_{1/2}^a = \bar{\Sigma} V_{-1}^a \quad (3.2.5)$$

with which the Yukawa couplings can be written as

$$< V_{-1}^{\mathbf{10}} V_{-1/2}^{\mathbf{16}} V_{-1/2}^{\mathbf{16}} > = < \bar{\Sigma}^2 (\mathbf{1}_2)_{-1} (\mathbf{8}_{-1}^v)_{-1} (\mathbf{8}_{-1}^v)_{-1} > . \quad (3.2.6)$$

The first obstacle any correlation function has to overcome, of course, is charge conservation, according to which the $\text{U}(1)$ charges must cancel in every factor. Now, according to (3.2.6), every coupling of the type (3.2.1) contains per construction the operator $\bar{\Sigma}^2$. We consider first the vectors. Starting with the vector A of Table 6 we need to check which pair of generations we can multiply to get something nonvanishing. Computing the charges of $(\bar{\Sigma}^2 A)$ results in

$$\begin{pmatrix} Q \\ \bar{Q} \end{pmatrix} (\bar{\Sigma}^2 A) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1/5 \\ 1/5 \end{pmatrix} \otimes \begin{pmatrix} 3/5 \\ 3/5 \end{pmatrix} \otimes \begin{pmatrix} 3/5 \\ 3/5 \end{pmatrix} \otimes \begin{pmatrix} 3/5 \\ 3/5 \end{pmatrix}, \quad (3.2.7)$$

hence pairs of generations are needed whose product leads to the negative of this charge array. Looking back at the generations we see that there are members of the pair families (I, II) on the

one hand, and (II,III) and (III, IV) on the other, which lead to the appropriate charges. Thus we have the following potentially nonvanishing Yukawa couplings

$$< A \cdot I \cdot II >, \quad < A \cdot II \cdot III >, \quad < A \cdot III \cdot IV > \quad (3.2.8)$$

for the simple reason that I – IV are the only generations with charge zero in the first factor (the only N=2 minimal factor affected by the simple current). Proceeding in the same manner one finds the remaining potentially nonvanishing couplings.

In order to compute the actual values of these couplings one uses the fact that a primary field in the N=2 minimal superconformal theory at level k , is just a product of a parafermionic field and a scalar field $\varphi(z)$

$$\Phi_{l,\bar{q},\bar{s}}^{l,q,s}(z, \bar{z}) = \phi_{l,\bar{q}-\bar{s}}^{l,q-s}(z, \bar{z}) e^{i(\alpha_{qs}\varphi(z) + \alpha_{\bar{q}\bar{s}}\varphi'(\bar{z}))}, \quad (3.2.9)$$

where

$$\alpha_{qs} = \frac{-q + \frac{s}{2}(k+2)}{\sqrt{k(k+2)}}. \quad (3.2.10)$$

The parafermionic field $\phi^{l,m}$ in turn can be expressed in terms of SU(2) primary fields and a further scalar field $\tilde{\varphi}$ as

$$\phi^{l,m} = G^{\frac{l}{2}, \frac{m}{2}} e^{-\frac{m}{2\sqrt{2}}\tilde{\varphi}}. \quad (3.2.11)$$

Thus the only nontrivial correlation functions one has to know in order to compute the Yukawa couplings of the superconformal model are the three-point functions of the SU(2) theory. These have been obtained by Zamolodchikov and Fateev [46], who found

$$< G_{j_1, \bar{m}_1}^{j_1, m_1} G_{j_1, \bar{m}_1}^{j_1, m_1} G_{j_1, \bar{m}_1}^{j_1, m_1} >_{\text{SU}_k} = \frac{\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ \bar{m}_1 & \bar{m}_2 & \bar{m}_3 \end{pmatrix} \rho_k(j_1, j_2, j_3)}{f(z_{12}\bar{z}_{12}, z_{13}\bar{z}_{13}, z_{23}\bar{z}_{23})}, \quad (3.2.12)$$

where $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ are the Wigner $3j$ -symbols

$$\begin{aligned} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} &= \sqrt{\frac{(j_1 + j_2 - j_3)!(j_1 - j_2 + j_3)!(-j_1 + j_2 + j_3)!}{(j_1 + j_2 + j_3 + 1)!}} \prod_{i=1}^3 (j_i + m_i) (j_i - m_i)! \times \\ &\times \sum_{z \in \mathbf{Z}} \frac{(-1)^{z+j_1-j_2-m_3}}{z!(j_1 + j_2 - j_3 - z)!(j_1 - m_1 - z)!(j_2 + m_2 - z)!(j_3 - j_2 + m_1 + z)!(j_3 - j_1 - m_2 + z)!} \end{aligned} \quad (3.2.13)$$

and

$$\rho_k(j_1, j_2, j_3) = F_k(j_1, j_2, j_3) \sqrt{\frac{\Gamma\left(\frac{k+3}{k+2}\right)}{\Gamma\left(\frac{k+1}{k+2}\right)} \prod_{r=1}^3 (2j_r + 1) \frac{\Gamma\left(1 - \frac{2j_r+1}{k+2}\right)}{\Gamma\left(1 + \frac{2j_r+1}{k+2}\right)}}, \quad (3.2.14)$$

where

$$F_k(j_1, j_2, j_3) = \frac{\pi_k(j_1 + j_2 + j_3 + 1) \pi_k(j_1 + j_2 - j_3) \pi_k(j_2 + j_3 - j_1) \pi_k(j_3 + j_1 - j_2)}{\pi_k(2j_1) \pi_k(2j_2) \pi_k(2j_3)} \quad (3.2.15)$$

with, finally,

$$\pi_k(j) = \prod_{r=1}^j \frac{\Gamma\left(1 + \frac{r}{k+2}\right)}{\Gamma\left(1 - \frac{r}{k+2}\right)}. \quad (3.2.16)$$

Plugging in these ingredients we can compute the necessary Yukawa couplings of the individual minimal theories. The relevant couplings for the level $k = 3$ theory are

$$\begin{aligned} \left\langle \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\rangle &= 1 & \left\langle \begin{bmatrix} 2 & -2 & 0 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\rangle &= 1 \\ \left\langle \begin{bmatrix} 3 & -3 & 0 \\ 3 & -3 & 0 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\rangle &= 1 & \left\langle \begin{bmatrix} 3 & -3 & 0 \\ 3 & -3 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \right\rangle &= 1 \\ \left\langle \begin{bmatrix} 2 & -1 & 0 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \right\rangle &= \kappa & \left\langle \begin{bmatrix} 2 & 3 & 1 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\rangle &= 1, \end{aligned} \quad (3.2.17)$$

where

$$\kappa = \left(\frac{\Gamma\left(\frac{3}{5}\right)^3 \Gamma\left(\frac{1}{5}\right)}{\Gamma\left(\frac{2}{5}\right)^3 \Gamma\left(\frac{4}{5}\right)} \right)^{1/2}. \quad (3.2.18)$$

With the couplings for the individual N=2 factors at hand we finally arrive at the Yukawa couplings of the full theory, the results of which are contained in Table 7.

10	Generations
A·	I·II = 1, II·III = 1, III·IV = 1
B·	I ² = 1, I·III = 1, II ² = κ^2 , II·III = κ , III ² = 1, III·IV = κ , IV ² = κ^2
C _i ·	I·V _i = 1, I·VI _i = 1, II·VI _i = κ , III·VI _i = 1, III·VII _i = κ
D _i ·	I·VII _i = 1, II·V _i = 1, II·VI _i = 1, III·VII _i = 1, IV·VI _i = 1
E _i ·	I·VI _i = 1, II·VI _i = κ , II·VII _i = κ^2 , III·V _i = 1, III·VI _i = 1, III·VII _i = κ , IV·VI _i = κ , IV·VII _i = κ^2
F _i ·	III·VI _i = κ , IV·VII _i = κ^3

Table 7: Yukawa couplings of the generations of the $SO(10)$ (80,0) daughter of $3^{\otimes 5}$.

4 Chiral ring and σ -model structure of the (80,0) theory

In this Section we determine the linear σ -model whose fixed point is described by the exactly solvable model analyzed in depth in the previous Section. This problem naturally splits into two

parts. The first is to determine the chiral ring of the (0,2) of the theory, defining a monomial algebra, the product structure of which captures the behaviour of the Yukawa couplings. In the context of (0,2) theories based either on Calabi–Yau manifolds or Landau–Ginzburg theories, this would appear to furnish the appropriate framework in which one eventually should understand the complete structure of the underlying string model, the reason being that Calabi–Yau manifolds are algebraic projective varieties and thus are embedded in a simple space, \mathbb{P}_n , although by a rather complicated map.

Being able to derive the ring structure from one or more superpotentials, in order to obtain a complete intersection, simplifies the analysis considerably, but does restrict the focus to a rather narrow framework, the confines of which has to be overcome eventually. In the context of (2,2) compactifications a well known technique for dealing with rings more general than those originating from complete intersections is furnished by toric geometry and an interesting problem for future work would be to formulate (0,2) theories based on toric varieties. Even toric geometry, however, is a rather restrictive framework and it would be of great interest to explore Calabi–Yau manifolds in purely algebraic, ring theoretic, terms.

In the present paper, however, our interest is a different one, and we focus on the particular exactly solvable theory we have been analyzing in some detail in the previous Section precisely because its massless sector is simplified by the fact that it does not contain antighenations. Because of this we might hope that its σ -model is described by a simple geometric structure. We will see below that this is indeed the case and that we are lead to a rank four vector bundle on a smooth codimension two complete intersection Calabi–Yau manifold. First we derive the chiral ring.

4.1 The (0,2) chiral ring of the (80,0) model

In order to extract the chiral ring we consider the detailed structure of the generations. Because the simple current acts only on the first factor of the tensor product of N=2 minimal theories, we expect the chiral primary fields of the unaffected minimal theories to remain unchanged. From the structure of the generations of type I – IV in Table 5 we indeed see that the basic field which appears in the last four minimal factors is the chiral primary field $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$. Thus we see that in order to relate these exact states to the complex coordinates of a ring we should make the identification

$$x_i \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (4.1.1)$$

where the nonzero state is in the $(i + 1)st$ factor of the 5 individual factors. From the remaining generation families V – VII of Table 5 we can then read off that the variables

$$\begin{aligned} y_1 &\sim \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ y_2 &\sim \begin{bmatrix} 2 & -3 & -1 \\ 2 & 2 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} -1 \\ 0 \end{bmatrix} \end{aligned} \quad (4.1.2)$$

must have degree $\deg(y_i) = 2$, in agreement with the anomalous weights of the states. It is important to note that for the ‘twisted’ state y_2 the contribution of the $U(1)_2$ is crucial in order for these fields to be chiral primary fields. Thus we have two types of coordinates x_i , $i = 1, 2, 3, 4$, coming from the inert minimal factors and y_l , $l = 1, 2$, generated by the simple current Υ . Using the relations (4.1.1) and (4.1.2) maps the generations into the monomial representations listed in Table 8.

Family	Monomial Representative	Degeneracy
I	$x_i^2 x_j^3$	12
II	$x_i x_j x_k^3$	12
III	$x_i x_j^2 x_k^2$	12
IV	$x_i x_j x_k x_l^2$	4
V_l	$x_i^3 y_l$	8
VI_l	$x_i x_j^2 y_l$	24
VII_l	$x_i x_j x_k y_l$	8

Table 8: Monomial representation of the generations of the $(80,0)$ model.

The chiral ring $\mathcal{R} = \frac{\mathbb{C}[x_i, y_l]}{\mathcal{I}}$ determined by these states is generated from free ring

$$\mathbb{C}[x_1, x_2, x_3, x_4, y_1, y_2] \quad (4.1.3)$$

by considering equivalence relations defined via the ideal

$$\mathcal{I}[x_1^4, x_2^4, x_3^4, x_4^4, y_1^2, y_2^2, y_1 y_2]. \quad (4.1.4)$$

The question then arises whether this ideal can be derived from a stable bundle over a Calabi–Yau threefold. We will show in the next Subsection that this is indeed possible. First, however, we complete our discussion of the spectrum by considering the vectors in $H^1(M, \Lambda^2 V)$, which can be identified with monomials of degree ten. There are a total of 72 of such elements, the explicit form of which is contained in Table 9.

Family	Monomial Representative	Degeneracy
A	$x_i x_j^3 x_k^3 x_l^3$	4
B	$x_i^2 x_j^2 x_k^3 x_l^3$	6
C_m	$y_m x_i^2 x_j^3 x_k^3$	24
D_m	$y_m x_i x_j x_k^3 x_l^3$	12
E_m	$y_m x_i x_j^2 x_k^2 x_l^3$	24
F_m	$y_m x_i^2 x_j^2 x_k^2 x_l^2$	2

Table 9: Monomial representation of the vectors of the $(80,0)$ model.

The precise relation of these monomial forms to the fields of the exactly solvable theory is indicated by the types A–F, referring back to Table 6. The two vectors G_1 , G_2 of Table 6 do not have a monomial representation. As we have seen in the preceding Section they also do not lead to nonvanishing Yukawa couplings. The product structure of the ring \mathcal{R} is then given by the nonvanishing relations described in Table 10.

Vector	Generations
A:	$I \cdot II, II \cdot III, III \cdot IV$
B:	$I^2, I \cdot III, II^2, II \cdot III, III^2, III \cdot IV, IV^2$
C_l :	$I \cdot V_l, I \cdot VI_l, II \cdot VI_l, III \cdot VI_l, III \cdot VII_l$
D_l :	$I \cdot VII_l, II \cdot V_l, II \cdot VI_l, III \cdot VII_l, IV \cdot VI_l$
E_l :	$I \cdot VI_l, II \cdot VI_l, II \cdot VII_l, III \cdot V_l, III \cdot VI_l, III \cdot VII_l, IV \cdot VI_l, IV \cdot VII_l$
F_l :	$III \cdot VI_l, IV \cdot VII_l$

Table 10: Product structure of the ring of the $(80,0)$ model.

Comparing the product structure of the chiral ring derived in Table 10 with the exactly solvable Yukawas of Table 7 shows that the simple renormalization

$$\begin{aligned}
II &\longrightarrow \kappa II, \quad IV \longrightarrow \kappa IV, \quad VII_l \longrightarrow \kappa VII_l, \\
A &\longrightarrow \kappa^{-1} A, \quad D_l \longrightarrow \kappa^{-1} D_l, \quad F_l \longrightarrow \kappa F_l
\end{aligned} \tag{4.1.5}$$

transforms one set of couplings into the other. It should be noted that in contradistinction to the situation for the theories discussed in the $(2,2)$ context [23, 24, 25, 9] the transformation is not fixed uniquely but is determined only up to a two parameter families of rescalings. If, e.g., we define $A \longrightarrow \kappa^a A$ and $VII \longrightarrow \kappa^x VII$, then all the remaining normalizations are determined by

the exponents a and x . In (4.1.5) we have set $a = -1$ and $x = 1$ which seemed to be the simplest choice because it minimizes the number of renormalizations. The fact that the map between the Yukawa couplings is not determined uniquely in the present context will turn out to be a generic feature even though the degree of nonuniqueness depends on the specific models, in particular their gauge groups. We therefore find that knowledge of the underlying exactly solvable theory is a slightly less powerful tool than it is in the context of (2,2) theories.

4.2 The (0,2) linear σ -model

At this point we have shown that there exists a chiral ring whose product structure is isomorphic to the Yukawa coupling structure of an exactly solvable theory. It is then natural to ask whether this ideal can be derived from the superpotential of a (0,2) σ -model. We will show that indeed it can. Recall [15, 33], that the essential structure of a (0,2) linear σ -model is encoded in the superpotentials $(W_r(\Phi_i), F_a^l(\Phi_i))$, where $W_r(\Phi_i)$ are polynomials of degree d_r which define the base space M of the vector bundle $V \longrightarrow M$ associated to the left-moving gauge fermions and the $F_a^l(\Phi_i)$ are polynomials of degree $\deg F_a^l = m_l - n_a$ which define the global structure of the bundle V . The appropriate constraints are imposed by introducing Lagrange multipliers Σ^r which are Fermi superfields with charge $(-d_r)$, as well as Lagrange multipliers Λ^a which are Fermi superfields with charges n_a . Finally, one introduces chiral superfields Θ_l with charges m_l such that $\sum_l m_l = \sum_a n_a$. The charges assigned to these fields read as follows

$$\begin{aligned} Q(\Phi_i) &= k_i, \quad i = 1, \dots, N_i; & Q(\Sigma^r) &= -d_r, \quad r = 1, \dots, N_r \\ Q(\Lambda^a) &= n_a, \quad a = 1, \dots, N_a = r + N_i; & Q(\Theta_l) &= -m_l, \quad l = 1, \dots, N_l. \end{aligned} \quad (4.2.1)$$

The action which summarizes the structure of the total bundle is then given by

$$\mathcal{A} = \int d^2z d\theta \left[\Sigma^r W_r(\Phi_i) + \Theta_l \Lambda^a F_a^l(\Phi_i) \right]. \quad (4.2.2)$$

The first term leads to constraints on the Φ_i to the effect that they take values on the hypersurface $W_r = 0$, whereas the second term ensures that the gauge fermions λ^a (i.e. the lowest components of the Λ^a) are sections of the bundle V .

The structure of the bundle V can be summarized concisely by the short exact sequence

$$0 \longrightarrow V \longrightarrow \bigoplus_{a=1}^{r+N_l} \mathcal{O}(n_a) \xrightarrow{F} \bigoplus_{l=1}^{N_l} \mathcal{O}(m_l) \longrightarrow 0 \quad (4.2.3)$$

which allows to compute the Chern class

$$c(V) = \frac{c\left(\bigoplus_{a=1}^{r+N_l} \mathcal{O}(n_a)\right)}{c\left(\bigoplus_{l=1}^{N_l} \mathcal{O}(m_l)\right)} \quad (4.2.4)$$

which, with

$$c\left(\bigoplus_{a=1}^{r+N_l} \mathcal{O}(n_a)\right) = \prod_a c(\mathcal{O}(n_a)) = \prod_a (1 + c_1(\mathcal{O}(n_a))) = \prod_a (1 + n_a h) \quad (4.2.5)$$

and

$$c\left(\bigoplus_{l=1}^{N_l} \mathcal{O}(m_l)\right) = \prod_l c(\mathcal{O}(m_l)) = \prod_l (1 + m_l h), \quad (4.2.6)$$

leads to

$$c(V) = \frac{\prod_a (1 + n_a h)}{\prod_l (1 + m_l h)}. \quad (4.2.7)$$

Expanding this expression leads to the individual Chern classes

$$\begin{aligned} c_1(V) &= \left[\sum_a n_a - \sum_l m_l \right] h \\ c_2(V) &= \frac{1}{2} \left[\sum_l m_l^2 - \sum_a n_a^2 \right] h^2 \\ c_3(V) &= -\frac{1}{3} \left[\sum_l m_l^3 - \sum_a n_a^3 \right] h^3. \end{aligned} \quad (4.2.8)$$

Recalling that for a complete intersection of the form $\mathbb{P}_{(k_1, \dots, k_{N_l})} [d_1 \cdots d_{N_r}]$ the second Chern class is given by $c_2(TM) = \frac{1}{2} [\sum_r d_r^2 - \sum_i k_i^2]$, the anomaly matching condition can be written as

$$\sum_l m_l^2 - \sum_a n_a^2 = \sum_r d_r^2 - \sum_i k_i^2. \quad (4.2.9)$$

Now, the structure of the ring \mathcal{R} given by (4.1.3) and (4.1.4) tells us that we should look for potentials (W_r, F_a^l) defining a bundle V of rank four which are of degree four. As derived in the previous Subsection the base space is spanned by four coordinates x_i , $i = 1, \dots, 4$ of unit weight and two coordinates y_l , $l = 1, 2$ of weight two. Since we wish to consider a 3-fold it follows that we need to impose two polynomials W_1, W_2 of degree four, i.e. the bundle V will live on the base configuration $M = \mathbb{P}_{(1,1,1,1,2,2)}[4 \ 4]$. This manifold is indeed a Calabi–Yau manifold, i.e. $c_1(M) = 0$, the Chern classes of which are $c_2(M) = 10h^2$ and $c_3(M) = -36h^3$, where h is the pullback of the generator of $H^2(M)$.

The simplest way to define a rank four bundle with potentials F_a of degree four is via the short sequence

$$0 \longrightarrow V \longrightarrow \bigoplus_{a=1}^5 \mathcal{O}(n_a = 1) \xrightarrow{F} \mathcal{O}(5) \longrightarrow 0, \quad (4.2.10)$$

the resulting bundle of which we denote by $V_{(1,1,1,1,1,5)}$, adopting the convention of [47]. The second Chern class of this bundle is equal to the second Chern class of the manifold and hence

the anomalies cancel. Furthermore $c_1(V) = 0$ and therefore the bundle

$$V_{(1,1,1,1,1;5)} \longrightarrow \mathbb{P}_{(1,1,1,1,2,2)}[4 \ 4] \quad (4.2.11)$$

describes the Calabi–Yau phase of the linear σ -model defined by the chiral primary fields with the charges $Q(\Phi_i) = 1$, $i = 1, \dots, 4$, $Q(\Phi_i) = 2$, $i = 5, 6$ and $Q(\Lambda^a) = 1$, $a = 1, \dots, 5$. With $c_3(V) = -160h^3$ one finds for the Euler number

$$\chi(V_{(1,1,1,1,1;5)}) = -160 \quad (4.2.12)$$

and because the manifold is smooth we expect that this theory has no antigerations. It follows from the structure of the chiral ring that this is indeed the case and therefore this σ -model leads to 80 generations, in agreement with the exact model discussed in Section 3.

The problem then boils down to finding a choice for the defining polynomials of the base and the defining data for the bundle $V_{(1,1,1,1,1;5)}$ which leads to the ideal (4.1.4). First one has to pick the defining data of the manifold, i.e. the polynomials W_r . A transverse choice is given by

$$\begin{aligned} W_1 &= \sum_i x_i^4 + \sum_l y_l^2 \\ W_2 &= \sum_i i x_i^4 + \sum_l l y_l^2. \end{aligned} \quad (4.2.13)$$

Next we need to pick the bundle data, i.e. the F_a s, which we do as

$$(F_a) = (x_1^4, x_2^4, x_3^4, x_4^4, y_1 y_2). \quad (4.2.14)$$

One easily checks that the ideal generated by (W_r, F_a) then is given by

$$\mathcal{I}[x_1^4, x_2^4, x_3^4, x_4^4, y_1^2, y_2^2, y_1 y_2]. \quad (4.2.15)$$

This is precisely the ideal we have derived previously from the structure of the generations in the exactly solvable theory.

Having succeeded in identifying the (0,2) Calabi–Yau manifold, the conformal fixed point of which is described by the exactly solvable (80,0) theory, we realize that the product structure we have computed in Section 4.1 for the chiral ring derived from the exact model is nothing but the product structure of the cohomology ring $H^p(M, \Lambda^q V)$. In particular the Yukawa couplings are determined by the product

$$H^1(M, V) \otimes H^1(M, V) \longrightarrow H^1(M, \Lambda^2 V), \quad (4.2.16)$$

where we have used the isomorphism $H^p(M, \Lambda^q V) = H^{D-p}(M, \Lambda^{r-q} V)$ for a bundle V of rank r on a D -dimensional variety.

The linear σ -model that we have thus derived from the exactly solvable model of Section 3 has been shown [33] to contain both a (0,2) Calabi–Yau as well as a Landau–Ginzburg phase. We therefore have provided for the $\text{SO}(10)$ –(80,0) (0,2)–theory all the ingredients familiar from the context of (2,2) exactly solvable Calabi–Yau σ -models. In particular we have established the existence of exactly solvable theories describing the conformal fixed points of (0,2) Calabi–Yau linear σ -models and have described its precise nature.

5 An $\text{SU}(5)$ (64,0) (0,2)–model derived from $3^{\otimes 5}$

5.1 The exactly solvable theory

The exact model is derived from the parent tensor theory $3^{\otimes 5}$ by inserting the two simple currents

$$\begin{aligned}\Upsilon_1 &= [3 \ 0 \ -1] \otimes [0 \ 0 \ 0] \otimes [0 \ 0 \ 0]^{\otimes 3} \otimes [1] \otimes [0] \otimes [0] \\ \Upsilon_2 &= [3 \ 0 \ -1] \otimes [3 \ 0 \ -1] \otimes [0 \ 0 \ 0]^{\otimes 3} \otimes [1] \otimes [1] \otimes [0].\end{aligned}\tag{5.1.1}$$

In this model the E_6 of the Gepner model is broken down to $\text{SU}(5)$ and hence the generation/antigeneration structure is determined by the **10** and $\overline{\mathbf{10}}$ representations of $\text{SU}(5)$. The massless spectrum of this model contains (among others) the modes of Table 11.

Representation	0	10	$\overline{\mathbf{10}}$	5	$\overline{\mathbf{5}}$
Spin 0	338	64	0	55	119
Spin 1	10	0	0	0	0

Table 11: *Massless spectrum of the (64,0) $\text{SU}(5)$ daughter of the $3^{\otimes 5}$ model.*

Now, recall that the **10** decomposes with respect to the maximal subgroup $\text{SO}(6) \times \text{U}(1)$ as

$$\mathbf{10} = \mathbf{6}_{-1} \oplus \mathbf{4}_{3/2}.\tag{5.1.2}$$

The generations that result are enumerated in Table 12, where we again use the abbreviations g_i of (3.1.3) for the two states appearing in those two minimal $\text{N}=2$ factors at level three that are affected by the simple current as well as the notation (3.1.4) for the $\text{U}(1)$ factors.

Type	Field	Number
I	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \end{bmatrix} u_0 u_0 \begin{bmatrix} v \\ v \end{bmatrix}$	6
II	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \end{bmatrix} u_0 u_0 \begin{bmatrix} v \\ v \end{bmatrix}$	3
III	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} u_0 u_0 \begin{bmatrix} v \\ v \end{bmatrix}$	3
IV _i	$g_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \end{bmatrix} u_i u_0 \begin{bmatrix} v \\ v \end{bmatrix}$	6
V _i	$g_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} u_i u_0 \begin{bmatrix} v \\ v \end{bmatrix}$	12
VI _i	$g_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} u_i u_0 \begin{bmatrix} v \\ v \end{bmatrix}$	2
VII _i	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} g_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \end{bmatrix} u_0 u_i \begin{bmatrix} v \\ v \end{bmatrix}$	6
VIII _i	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} g_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} u_0 u_i \begin{bmatrix} v \\ v \end{bmatrix}$	12
IX _i	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} g_i \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} u_0 u_i \begin{bmatrix} v \\ v \end{bmatrix}$	2
X _{ij}	$g_i g_j \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} u_i u_j \begin{bmatrix} v \\ v \end{bmatrix}$	12

Table 12: Generations of the $SU(5)$ $(64,0)$ -daughter of $3^{\otimes 5}$.

The **5**s of the $SU(5)$ decompose with respect to the maximal subgroup $SO(6) \times U(1)$ as

$$\mathbf{5} = \mathbf{4}_{-1/2} \oplus \mathbf{1}_2. \quad (5.1.3)$$

Counting the singlet part and recalling the abbreviations (3.1.5) the 55 representations of the **5** then take the form presented in Table 13.

Type	Field	Number
A _i	$v_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ 2 & -4 & -2 \end{bmatrix} \begin{bmatrix} 3 & -3 & 0 \\ 3 & -5 & -2 \end{bmatrix} \begin{bmatrix} 3 & -3 & 0 \\ 3 & -5 & -2 \end{bmatrix} u_i^+ u_0 \begin{bmatrix} 0 \\ v \end{bmatrix}$	6
B _i	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & -2 \end{bmatrix} v_i \begin{bmatrix} 2 & -2 & 0 \\ 2 & -4 & -2 \end{bmatrix} \begin{bmatrix} 3 & -3 & 0 \\ 3 & -5 & -2 \end{bmatrix} \begin{bmatrix} 3 & -3 & 0 \\ 3 & -5 & -2 \end{bmatrix} u_0 u_i^+ \begin{bmatrix} 0 \\ v \end{bmatrix}$	6
C _{ij}	$v_i v_j \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} 3 & -3 & 0 \\ 3 & -5 & -2 \end{bmatrix} \begin{bmatrix} 3 & -3 & 0 \\ 3 & -5 & -2 \end{bmatrix} u_i^+ u_j^+ \begin{bmatrix} 0 \\ v \end{bmatrix}$	12
D _{ij}	$v_i v_j \begin{bmatrix} 3 & -3 & 0 \\ 3 & -5 & -2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ 2 & -4 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & -3 & -2 \end{bmatrix} u_i^+ u_j^+ \begin{bmatrix} 0 \\ v \end{bmatrix}$	24
E _{ij}	$v_i v_j \begin{bmatrix} 2 & -2 & 0 \\ 2 & -4 & -2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ 2 & -4 & -2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ 2 & -4 & -2 \end{bmatrix} u_i^+ u_j^+ \begin{bmatrix} 0 \\ v \end{bmatrix}$	4
F	$\begin{bmatrix} 3 & -4 & -1 \\ 3 & -5 & -2 \end{bmatrix} \begin{bmatrix} 3 & -4 & -1 \\ 3 & -5 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 1 & -3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 1 & -3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & -4 & -2 \end{bmatrix} u_1^+ u_1^+ \begin{bmatrix} 0 \\ v \end{bmatrix}$	3

Table 13: The **5**s of the $SU(5)$ $(64,0)$ $(0,2)$ model of 3^5 .

We now proceed to the couplings.

5.2 The exact Yukawa couplings

Recalling the decomposition $\mathbf{10} = \mathbf{6}_{-1} \oplus \mathbf{4}_{3/2}$ one notes that the allowed nonvanishing combination for the Yukawa couplings can be taken to be of the form

$$< \mathbf{5} \cdot \mathbf{10} \cdot \mathbf{10} > = < \mathbf{1}_2 \cdot \mathbf{6}_{-1} \cdot \mathbf{6}_{-1} > . \quad (5.2.1)$$

Getting the pictures aligned again involves $\bar{\Sigma} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{\otimes 5}$, and following our discussion of the first example we find that the only nonvanishing Yukawa couplings involving, say, the field A, are given by

$$< A_1 \cdot \text{I} \cdot \text{IV}_1 >, \quad < A_1 \cdot \text{I} \cdot \text{V}_1 >, \quad < A_1 \cdot \text{II} \cdot \text{V}_1 >, \quad < A_1 \cdot \text{III} \cdot \text{V}_1 >, \quad < A_1 \cdot \text{III} \cdot \text{VI}_1 > . \quad (5.2.2)$$

For the explicit evaluation of these couplings it again suffices to consider the basic couplings in the individual N=2, $k = 3$ minimal theory collected in (3.2.17). Proceeding in this manner one finds again all possibly nonvanishing Yukawas. The result is summarized in Table 14.

5	Generations
$A_i \cdot$	$\text{I} \cdot \text{IV}_i = 1, \text{I} \cdot \text{V}_i = 1, \text{II} \cdot \text{V}_i = \kappa, \text{III} \cdot \text{V}_i = 1, \text{III} \cdot \text{VI}_i = \kappa$
$B_i \cdot$	$\text{I} \cdot \text{VII}_i = 1, \text{I} \cdot \text{VIII}_i = 1, \text{II} \cdot \text{VIII}_i = \kappa, \text{III} \cdot \text{VIII}_i = 1, \text{III} \cdot \text{IX}_i = \kappa$
$C_{ij} \cdot$	$\text{I} \cdot \text{X}_{ij} = 1, \text{IV}_i \cdot \text{VII}_j = 1, \text{V}_i \cdot \text{VIII}_j = 1$
$D_{ij} \cdot$	$\text{I} \cdot \text{X}_{ij} = 1, \text{II} \cdot \text{X}_{ij} = \kappa, \text{III} \cdot \text{X}_{ij} = 1, \text{IV}_i \cdot \text{VIII}_j = 1, \text{V}_i \cdot \text{VII}_j = 1, \text{V}_i \cdot \text{IX}_j = \kappa, \text{VI}_i \cdot \text{VIII}_j = \kappa$
$E_{ij} \cdot$	$\text{III} \cdot \text{X}_{ij} = \kappa, \text{V}_i \cdot \text{VIII}_j = \kappa, \text{VI}_i \cdot \text{IX}_j = \kappa^3$

Table 14: *Yukawa couplings of the (64,0) SU(5) theory.*

As indicated in Table 14 there are no couplings involving the modes F_i .

5.3 The chiral ring of the (64,0) SU(5) model: states and product structure

The structure of the chiral ring for the present model can be derived in a way completely analogous to our discussion of the first example. Consider again the structure of those generations that do not contain contributions from the minimal factors affected by the simple current. These suggest that again we should introduce coordinates x_i related to the exact state

$$x_i \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (5.3.1)$$

where the nonzero state is in the $(i+2)$ st factor of the n individual factors. Again we can read off from the structure of the generations that the weight of the corresponding fields is unity. From the remaining generations we can then read off that the variables

$$\begin{aligned}
y_1 &\sim \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
y_2 &\sim \begin{bmatrix} 2 & -3 & -1 \\ 2 & 2 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} -1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
z_1 &\sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
z_2 &\sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 2 & -3 & -1 \\ 2 & 2 & 0 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} -1 \\ 0 \end{bmatrix}
\end{aligned} \tag{5.3.2}$$

must have degree $\deg(y_i) = 2 = \deg(z_i)$. With these coordinates the generations described in Subsection 5.1 are mapped into the monomials of Table 15.

Family	Monomial Representative	Degeneracy
I	$x_i^2 x_j^3$	6
II	$x_i x_j x_k^3$	3
III	$x_i x_j^2 x_k^2$	3
IV _l	$y_l x_i^3$	6
V _l	$y_l x_i x_j^2$	12
VI _l	$y_l x_i x_j x_k$	2
VII _m	$z_m x_i^3$	6
VIII _m	$z_m x_i x_j^2$	12
IX _m	$z_m x_i x_j x_k$	2
X _{l,m}	$y_l z_m x_i$	12

Table 15: Monomial representation of the generations of the $(64, 0)$ model.

It follows from the form of the generations that the ideal \mathcal{I} which generates the chiral ring $\mathcal{R} = \frac{\mathbb{C}[x_i, y_j, z_k]}{\mathcal{I}}$ from the free ring $\mathbb{C}[x_i, y_j, z_k]$ is generated by

$$\begin{aligned}
x_i^4 &= 0, \quad i = 1, 2, 3 \\
y_l y_{l'} &= 0, \quad l, l' = 1, 2 \\
z_m z_{m'} &= 0, \quad m, m' = 1, 2.
\end{aligned} \tag{5.3.3}$$

Next we consider the **5** representation, i.e. the monomials of degree ten, which take the form summarized in Table 16.

Family	Monomial Representative	Degeneracy
A_l	$y_l x_i^2 x_j^3 x_k^3$	6
B_m	$z_m x_i^2 x_j^3 x_k^3$	6
$C_{l,m}$	$y_l z_m x_i^3 x_j^3$	12
$D_{l,m}$	$y_l z_m x_i x_j^2 x_k^3$	24
$E_{l,m}$	$y_l z_m x_i^2 x_j^2 x_k^2$	4

Table 16: Monomial representation of the vectors of the $(64,0)$ model.

Thus there are a total of 52 vectors which can be represented by monomials. It is clear from their structure that for all generations $\gamma_i^2 \in \mathcal{I}$, where \mathcal{I} is the ideal described above. Thus there exists no nonvanishing Yukawa ‘selfcoupling’ of any generation. The Yukawas follow from

$$\gamma_i \cdot \gamma_j \sim v_{ij}, \quad (5.3.4)$$

where v_{ij} is any of the **5s**. With this one finds the nonvanishing couplings of Table 17.

Vector	Generations
A_l :	$I \cdot IV_l, I \cdot V_l, II \cdot V_l, III \cdot V_l, III \cdot VI_l$
B_l :	$I \cdot VII_l, I \cdot VIII_l, II \cdot VIII_l, III \cdot VIII_l, III \cdot IX_l$
C_{lm} :	$I \cdot X_{lm}, IV_l \cdot VII_m, V_l \cdot VIII_m$
D_{lm} :	$I \cdot X_{lm}, II \cdot X_{lm}, III \cdot X_{lm}, IV_l \cdot VIII_m, V_l \cdot VII_m, V_l \cdot IX_m, VI_l \cdot VIII_m$
E_{lm} :	$III \cdot X_{lm}, V_l \cdot VIII_m, VI_l \cdot IX_m$

Table 17: Product structure of the chiral ring of the $(64,0)$ $SU(5)$ theory.

Renormalizing the fields as

$$II \longrightarrow \kappa II, \quad VI_l \longrightarrow \kappa VI_l, \quad IX_l \longrightarrow \kappa IX_l, \quad E_{lm} \longrightarrow \kappa E_{lm} \quad (5.3.5)$$

then transforms the sigma model couplings into the exactly solvable couplings. As in the $(80,0)$ model this transformation is not unique, the difference being that now we have three scales at our disposal.

5.4 Remarks concerning the linear σ -model

As in the previous Section we wish to identify superpotentials $(W_r(\Phi_i), F_a^l(\Phi_i))$ which in the present case define a rank five bundle over a base space spanned by coordinates (x_i, y_l, z_m) , $i = 1, 2, 3$; $l =$

1, 2; $m = 1, 2$. The structure of the chiral ring suggests that we again consider potentials W_r of degree $d_r = 4$. This does not seem to work however because we have seven coordinates of weights k_i such that the sum of the weights is eleven. Therefore it appears that in order to define a Calabi–Yau threefold we have to introduce three polynomials W_r of degree d_r such that $\sum d_r = 11$. This leads one to introduce a further coordinate of weight five and to consider the configuration

$$\mathbb{P}_{(1,1,1,2,2,2,2,5)}[4\ 4\ 4\ 4]. \quad (5.4.1)$$

The virtue of this space is that it has a number of pleasant properties. First, it satisfies the anomaly condition

$$\sum_i k_i = \sum_r d_r, \quad (5.4.2)$$

thereby defining a complete intersection Calabi–Yau manifold of codimension 4. Second, it satisfies a condition familiar from the context of Landau–Ginzburg theories [27]. Namely the total charge of the theory is the codimension of the corresponding σ -model. Third, it does not have any orbifold singularities. This is an expected feature if the Kähler sector is to be absent for a (0,2) Calabi–Yau manifold. The disadvantage is that this space does not allow for a transverse choice of polynomials. It has a hypersurface singularity at one point. We hope that future insight into the resolution of this singularity will in fact turn this apparent difficulty into a virtue which will resolve a second puzzle which this space generates.

As in the theory discussed in Sections 3 and 4 we have been lead to a ring which is generated by an ideal spanned by elements of degree four. All that has happened is that the number of generators has increased, a change which we have already incorporated in our choice of the base space. It thus would appear natural to consider a bundle with the same quantum numbers of the gauge fermions as before. This leads us to

$$V_{(1,1,1,1,1,5)} \longrightarrow \mathbb{P}_{(1,1,1,2,2,2,2,5)}[4\ 4\ 4\ 4]. \quad (5.4.3)$$

Furthermore, the condition involving the second Chern classes of the gauge vector bundle and the tangent bundle is met by this choice of a bundle and therefore the linear σ -model defined by this structure satisfies all of the anomaly conditions. If the bundle would be defined over a smooth manifold however, it would be of rank four and not of rank five. We have to leave for future work whether a better understanding of the resolution does in fact cancel these two difficulties and make this bundle into a proper (0,2) Calabi–Yau manifold or whether an altogether different bundle can be found.

6 The (50,0) $SU(3) \times SU(2)$ (0,2)–model derived from $3^{\otimes 5}$

6.1 The exact theory

Consider the quintic tensor model $3^{\otimes 5}$ enhanced with the three simple currents

$$\begin{aligned}\Upsilon_1 &= [3 \ 0 \ -1] \otimes [0 \ 0 \ 0] \otimes [0 \ 0 \ 0] \otimes [0 \ 0 \ 0]^{\otimes 2} \otimes [1] \otimes [0] \otimes [0] \otimes [0] \\ \Upsilon_2 &= [3 \ 0 \ -1] \otimes [3 \ 0 \ -1] \otimes [0 \ 0 \ 0] \otimes [0 \ 0 \ 0]^{\otimes 2} \otimes [1] \otimes [1] \otimes [0] \otimes [0] \\ \Upsilon_3 &= [3 \ 0 \ -1] \otimes [3 \ 0 \ -1] \otimes [3 \ 0 \ -1] \otimes [0 \ 0 \ 0]^{\otimes 2} \otimes [1] \otimes [1] \otimes [1] \otimes [0].\end{aligned}\quad (6.1.1)$$

In this model the E_6 of the parent Gepner model is broken to $SU(3) \times SU(2)$, hence the modes are arranged in representations of $SU(3) \times SU(2)$ as shown for the massless sector in Table 18.

Representation:	0	2	3	$\bar{3}$	6	$\bar{6}$
Spin 0:	370	134	54	154	50	0
Spin 1:	13	0	0	0	0	0

Table 18: *Massless spectrum of the $(50,0)$ $SU(3) \times SU(2)$ model.*

The form of the generations can be found in Table 19.

Type	Field	Number
I	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \end{bmatrix} u_0^3 \begin{bmatrix} v \\ v \end{bmatrix}$	2
II _i	$g_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \end{bmatrix} u_i u_0^2 \begin{bmatrix} v \\ v \end{bmatrix}$	4
III _i	$g_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} u_i u_0^2 \begin{bmatrix} v \\ v \end{bmatrix}$	4
IV _i	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} g_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \end{bmatrix} u_0 u_i u_0 \begin{bmatrix} v \\ v \end{bmatrix}$	4
V _i	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} g_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} u_0 u_i u_0 \begin{bmatrix} v \\ v \end{bmatrix}$	4
VI _i	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} g_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \end{bmatrix} u_0^2 u_i \begin{bmatrix} v \\ v \end{bmatrix}$	4
VII _i	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} g_i \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} u_0^2 u_i \begin{bmatrix} v \\ v \end{bmatrix}$	4
VIII _{ij}	$g_i g_j \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} u_i u_j u_0 \begin{bmatrix} v \\ v \end{bmatrix}$	8
IX _{ij}	$g_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} g_j \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} u_i u_0 u_j \begin{bmatrix} v \\ v \end{bmatrix}$	8
X _{ij}	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} g_i g_j \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} u_0 u_i u_j \begin{bmatrix} v \\ v \end{bmatrix}$	8

Table 19: *Generations of the $(50,0)$ $SU(3) \times SU(2)$ daughter of the $3^{\otimes 5}$ model.*

whereas the **3**s can be found in Table 20 in which we have used the abbreviations

$$s_0 = \begin{bmatrix} 2 & 2 & 0 \\ 2 & -4 & -2 \end{bmatrix}, \quad s_1 = \begin{bmatrix} 2 & -3 & -1 \\ 2 & -4 & -2 \end{bmatrix},$$

$$t_0 = \begin{bmatrix} 1 & 3 & 2 \\ & -3 & -2 \end{bmatrix}, \quad t_1 = \begin{bmatrix} 1 & -2 & -1 \\ & -3 & -2 \end{bmatrix}. \quad (6.1.2)$$

Note that the s_i s are just the v_i s with conjugate charge on the left side.

Type	Field	Number
A_{ij}	$v_i v_j \begin{bmatrix} 0 & 0 & 0 \\ & -2 & -2 \end{bmatrix} \begin{bmatrix} 3 & -3 & 0 \\ & -5 & -2 \end{bmatrix} \begin{bmatrix} 3 & -3 & 0 \\ & -5 & -2 \end{bmatrix} u_i^+ u_j^+ u_0 \begin{bmatrix} 0 \\ v \end{bmatrix}$	4
B_{ij}	$v_i \begin{bmatrix} 0 & 0 & 0 \\ & -2 & -2 \end{bmatrix} v_j \begin{bmatrix} 3 & -3 & 0 \\ & -5 & -2 \end{bmatrix} \begin{bmatrix} 3 & -3 & 0 \\ & -5 & -2 \end{bmatrix} u_i^+ u_0 u_j^+ \begin{bmatrix} 0 \\ v \end{bmatrix}$	4
C_{ij}	$\begin{bmatrix} 0 & 0 & 0 \\ & -2 & -2 \end{bmatrix} v_i v_j \begin{bmatrix} 3 & -3 & 0 \\ & -5 & -2 \end{bmatrix} \begin{bmatrix} 3 & -3 & 0 \\ & -5 & -2 \end{bmatrix} u_0 u_i^+ u_j^+ \begin{bmatrix} 0 \\ v \end{bmatrix}$	4
D_{ijk}	$v_i v_j v_k \begin{bmatrix} 1 & -1 & 0 \\ & -3 & -2 \end{bmatrix} \begin{bmatrix} 3 & -3 & 0 \\ & -5 & -2 \end{bmatrix} u_i^+ u_j^+ u_k^+ \begin{bmatrix} 0 \\ v \end{bmatrix}$	16
E_{ijk}	$v_i v_j v_k \begin{bmatrix} 2 & -2 & 0 \\ & -4 & -2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ & -4 & -2 \end{bmatrix} u_i^+ u_j^+ u_k^+ \begin{bmatrix} 0 \\ v \end{bmatrix}$	8
F_i	$t_i \begin{bmatrix} 3 & -4 & -1 \\ & -5 & -2 \end{bmatrix} \begin{bmatrix} 3 & -4 & -1 \\ & -5 & -2 \end{bmatrix} \begin{bmatrix} 1 & -3 & -2 \\ & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ & 2 & 0 \end{bmatrix} u_i^+ u_1^{+2} \begin{bmatrix} 0 \\ v \end{bmatrix}$	4
G_i	$\begin{bmatrix} 3 & -4 & -1 \\ & -5 & -2 \end{bmatrix} t_i \begin{bmatrix} 3 & -4 & -1 \\ & -5 & -2 \end{bmatrix} \begin{bmatrix} 1 & -3 & -2 \\ & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ & 2 & 0 \end{bmatrix} u_1^+ u_i^+ u_1^+ \begin{bmatrix} 0 \\ v \end{bmatrix}$	4
H_i	$\begin{bmatrix} 3 & -4 & -1 \\ & -5 & -2 \end{bmatrix} \begin{bmatrix} 3 & -4 & -1 \\ & -5 & -2 \end{bmatrix} t_i \begin{bmatrix} 1 & -3 & -2 \\ & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ & 2 & 0 \end{bmatrix} u_1^{+2} u_i^+ \begin{bmatrix} 0 \\ v \end{bmatrix}$	4
I_i	$s_i \begin{bmatrix} 3 & -4 & -1 \\ & -5 & -2 \end{bmatrix} \begin{bmatrix} 3 & -4 & -1 \\ & -5 & -2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ & 2 & 0 \end{bmatrix} u_i^+ u_1^{+2} \begin{bmatrix} 0 \\ v \end{bmatrix}$	2
J_i	$\begin{bmatrix} 3 & -4 & -1 \\ & -5 & -2 \end{bmatrix} s_i \begin{bmatrix} 3 & -4 & -1 \\ & -5 & -2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ & 2 & 0 \end{bmatrix} u_1^+ u_i^+ u_1^+ \begin{bmatrix} 0 \\ v \end{bmatrix}$	2
K_i	$\begin{bmatrix} 3 & -4 & -1 \\ & -5 & -2 \end{bmatrix} \begin{bmatrix} 3 & -4 & -1 \\ & -5 & -2 \end{bmatrix} s_i \begin{bmatrix} 2 & -2 & 0 \\ & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ & 2 & 0 \end{bmatrix} u_1^{+2} u_i^+ \begin{bmatrix} 0 \\ v \end{bmatrix}$	2

Table 20: The $\mathbf{3}$ s of the $(50,0)$ $SU(3) \times SU(2)$ daughter of the $3^{\otimes 5}$ model

6.2 Exact Yukawa couplings

Due to the following decomposition of the representations of E_3 in those of $SO(4) \times U(1)$

$$\begin{aligned} \mathbf{6} &= (\mathbf{3}, \mathbf{2}) = \mathbf{4}_{-1}^v \oplus \mathbf{2}_2^c \\ \mathbf{3} &= (\mathbf{3}, \mathbf{0}) = \mathbf{1}_2 \oplus \mathbf{2}_{-1}^s \end{aligned} \quad (6.2.1)$$

we can calculate the Yukawa coupling in the following way

$$< \mathbf{3} \mathbf{6} \mathbf{6} > = < \mathbf{1}_2 \mathbf{4}_{-1}^v \mathbf{4}_{-1}^v > . \quad (6.2.2)$$

Proceeding in a completely analogous way as in the two former examples leads to the nonvanishing Yukawas of Table 21.

3	Generations
A_{ij}	$I \cdot VIII_{ij} = 1, II_i \cdot IV_j = 1, III_i \cdot V_j = 1$
B_{ij}	$I \cdot IX_{ij} = 1, II_i \cdot VI_j = 1, III_i \cdot VII_j = 1$
C_{ij}	$I \cdot X_{ij} = 1, IV_i \cdot VI_j = 1, V_i \cdot VII_j = 1$
D_{ijk}	$II_i \cdot X_{jk} = 1, III_i \cdot X_{jk} = 1, IV_j \cdot IX_{ik} = 1, V_j \cdot IX_{ik} = 1, VI_k \cdot VIII_{ij} = 1, VII_k \cdot VIII_{ij} = 1$
E_{ijk}	$III_i \cdot X_{jk} = \kappa, V_j \cdot IX_{ik} = \kappa, VII_k \cdot VIII_{ij} = \kappa,$

Table 21: *Yukawa couplings of the generations of the $SU(3) \times SU(2)$ (50,0) daughter of $3^{\otimes 5}$.*

6.3 Extracting the chiral ring

As is familiar by now, the structure of the chiral primary ring can be read off from the form of the generations top down. It is clear from the structure of the simple current that the weights of the chiral fields of the last two minimal factors in the tensor model are not changed. Thus we are led to introduce the coordinates of Table 22.

Coordinate	Weight
$x_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	1
$x_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	1
$y_1 = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	2
$y_2 = \begin{bmatrix} 2 & -3 & -1 \\ 2 & 2 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} -1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	2
$z_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	2
$z_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 2 & -3 & -1 \\ 2 & 2 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} -1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	2
$w_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	2
$w_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 2 & -3 & -1 \\ 2 & 2 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} -1 \\ 0 \end{bmatrix}$	2

Table 22: *Coordinates of the monomial ring for the (50,0) model.*

The generations then take the form shown in Table 23.

Family	Monomial Representative	Degeneracy
I	$x_i^2 x_j^3$	2
II _l	$u_l x_i^3$	4
III _l	$u_l x_i x_j^2$	4
IV _m	$v_m x_i^3$	4
V _m	$v_m x_i x_j^2$	4
VI _n	$w_n x_i^3$	4
VII _n	$w_n x_i x_j^2$	4
VIII _{lm}	$u_l v_m x_i$	8
IX _{ln}	$u_l w_n x_i$	8
X _{mn}	$v_m w_n x_i$	8

Table 23: *Monomial representation of the generations of the (50,0) model.*

From the generations we read off the ideal

$$\mathcal{I}[x_i^4, u_a u_b, v_a v_b, w_a w_b], \quad (6.3.1)$$

from which we see in turn that the monomials of degree ten decompose into the types listed in Table 24.

Family	Monomial Representative	Degeneracy
A _{lm}	$u_l v_m x_1^3 x_2^3$	4
B _{ln}	$u_l w_n x_1^3 x_2^3$	4
C _{mn}	$v_m w_n x_1^3 x_2^3$	4
D _{lmn}	$u_l v_m w_n x_i x_j^3$	16
E _{lmn}	$u_l v_m w_n x_i^2 x_j^2$	8

Table 24: *Monomial representation of the **3**s of the (50,0) model.*

Thus we find that in the present case only 36 of the total of 54 **3**s admit a monomial representation. Comparing the product structure of the chiral ring with the exact Yukawa couplings we again find complete agreement using the renormalization

$$E \longrightarrow \kappa E. \quad (6.3.2)$$

For this model we have the least predictive power of the three exactly solvable theory we have discussed. As in the first two examples the number of undetermined scalings is given by $(\text{rk}(V)-2)$ and therefore we find that we can adjust four different renormalizations.

6.4 Remarks concerning a possible σ -model representation

The story for the σ -model of the (50,0) model follows the pattern developed in the previous Sections for the (80,0) and the (64,0) models. The additional simple current has led to a further pair of coordinates of weight two. Again the ideal is generated by monomials of weight four and therefore we are led to consider the base space

$$\mathbb{P}_{(1,1,2,2,2,2,2,2,5,5)}[4\ 4\ 4\ 4\ 4\ 4], \quad (6.4.1)$$

where we have added two coordinates of weight five for reasons described already in our discussion of the (64,0) model in Section 5. Again this space does not admit transverse polynomials and will have hypersurface singularities, which in the present example is a complex curve, the projective line. Computing the second Chern class, we are led to consider yet again the same vector bundle

$$V_{(1,1,1,1,1,5)} \longrightarrow \mathbb{P}_{(1,1,2,2,2,2,2,2,5,5)}[4\ 4\ 4\ 4\ 4\ 4], \quad (6.4.2)$$

which presumably is modified by the resolution of the real-dimension two singular set in such a way as to lead to a rank six bundle.

Comparing the σ -models of all three models discussed in this paper the emerging pattern indicates that a hypersurface singular set of real-dimension d should increase the rank of the bundle to $(4 + d)$.

7 Conclusion

In the present paper we have established by construction the existence of conformal fixed points of (0,2) Calabi–Yau σ -models. Along the way we have identified the exactly solvable nature of such a conformal fixed point for the particular framework of (0,2) σ -models discussed in [33]. This generalizes to the framework of (0,2) compactification the work of [13, 14, 15] on the triality of exactly solvable models, Landau–Ginzburg theories and Calabi–Yau manifolds in the context of (2,2) compactifications. Our result thus unifies the different description of (0,2) vacua in a similar manner and allows us to call on many different techniques which are available in conformal field theory, the framework of (non)linear σ -models and the theory of stable vector bundles over algebraic varieties, in order to attack some of the outstanding questions mentioned in the introduction,

in particular possible generalizations of mirror symmetry and the structure of the $(0,2)$ moduli space.

There are a number of avenues that present themselves for further exploration. First, it is important to understand the map between the chiral primary fields of the exact theory and the chiral rings in a more systematic fashion than we have described here. Simple currents behave like orbifolds in many respects, which raises the hope that the analysis of [18] can be generalized to the framework of $(0,2)$ theories. Even without such an understanding it would be of interest to establish more $(0,2)$ triality relations even in the context of the type of exactly solvable $(0,2)$ theories which we have discussed. Instead of modifying Gepner models with simple currents, one can also start with more general classes of $N=2$ superconformal field theories, such as Kazama–Suzuki models and construct $(0,2)$ models by modifying their modular invariants with appropriate simple currents.

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